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LETTER TO THE EDITOR

# Critical behaviour in parabolic geometries

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**Abstract.** We study two-dimensional systems with boundary curves described by power laws. Using conformal mappings we obtain the correlations at the bulk critical point. Three different classes of behaviour are found and explained by scaling arguments which also apply to higher dimensions. For an Ising system of parabolic shape the behaviour of the order at the tip is also found.

The shape of a system undergoing a second-order phase transition can have a strong influence on its critical behaviour. This is shown by the results for edges (in 3D) or corners (in 2D). The local critical exponents are then continuous functions of the corresponding angle [1–6]. But this striking feature also raises the question which property of the boundary actually causes it and what would be obtained for other shapes. For the critical behaviour long-range effects are essential and thus a simple rounding of the corner will not matter [7]. We therefore study here shapes which differ from the corner geometry in the large: the boundary curves are described by power laws and do not have asymptotes. The prototype is the parabola. We use conformal mappings to obtain the critical correlation functions for various two-dimensional geometries. When the system forms the interior of a general parabolic figure, we find a new unusual form of the critical behaviour. When the system forms the exterior, on the other hand, one recovers the behaviour of a system with either a straight surface or a cut. These results can be understood from the way the boundary curves behave under renormalization. A similar classification will therefore hold in three dimensions. Our results at the critical point are complemented by a calculation of the tip magnetization for an Ising model of parabolic shape which also shows unusual features.

Consider first a system with free boundaries in the form of a simple parabola $v^2 = 2pu + p^2$ in the plane $w = u + \mathrm{i}v$ as in figure 1($a$). It can be related to the half-plane $z = x + \mathrm{i}y$, $y > 0$ by the conformal map

$$z = \mathrm{i}\cosh\left(\pi\sqrt{\frac{w}{2p}}\right). \tag{1}$$

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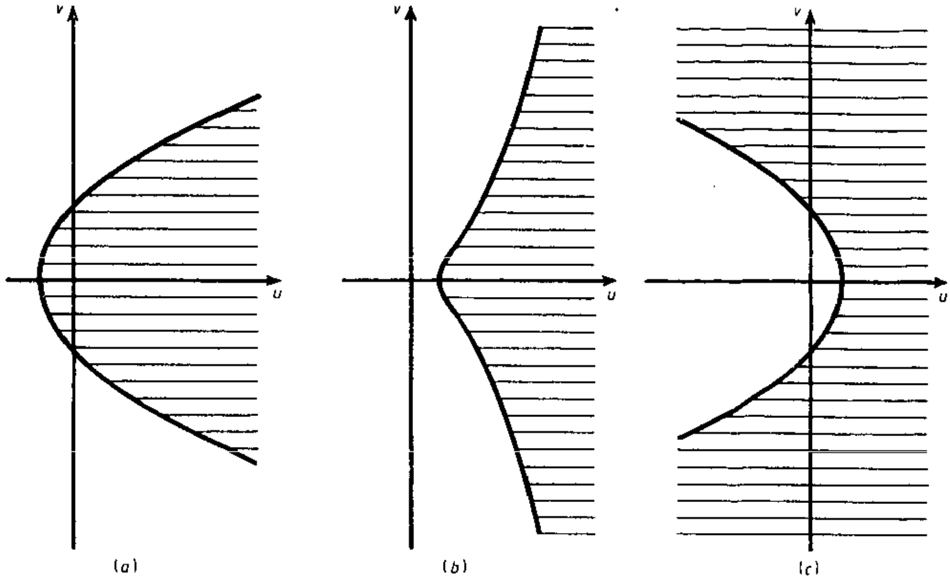


**Figure 1.** The three types of geometries considered in the text.

At criticality, the correlation function in the half-plane has the form [3]

$$G(z_1, z_2) = (y_1 y_2)^{-x} \psi(\omega) \tag{2}$$

where the scaling function depends on the variable $\omega = 4y_1y_2/ \mid z_1 - z_2 \mid^2$ and has the asymptotic form $\psi(\omega) \sim \omega^{x_1}$ for small $\omega$. Here $x$ and $x_1$ denote the bulk and surface scaling dimensions, respectively, of the operators in $G$. Using the standard transformation [8], one finds in the parabolic geometry

$$G(w_1, w_2) = \lambda_1^x \lambda_2^x (\cosh\zeta_1 \cosh\zeta_2 \cos\eta_1 \cos\eta_2)^{-x} \psi(\omega) \tag{3}$$

where the rescaling factors are

$$\lambda_i = \frac{\pi^2}{4p} \left( \frac{\sinh^2 \zeta_i + \sin^2 \eta_i}{\zeta_i^2 + \eta_i^2} \right)^{1/2} \tag{4}$$

and we have used parabolic coordinates $\pi\sqrt{w/2p} = \zeta + i\eta$. The system is then characterized by $0 \leqslant \eta \leqslant \pi/2$. If $w_1$, $w_2$ lie on the positive $u$-axis, the variable $\omega$ is given by $\omega = 4\cosh\zeta_1 \cosh\zeta_2/(\cosh\zeta_1 - \cosh\zeta_2)^2$. One then finds for $\zeta_1$ fixed and $\zeta_2 \gg 1$ $(u_2 \gg p)$

$$G(w_1, w_2) = A(u_1) \frac{1}{u_2^{x/2}} \exp\left(-\pi x_1 \sqrt{\frac{u_2}{2p}}\right). \tag{5}$$

This is not a simple power law in $u_2$ as one finds for a corner, which clearly shows the difference between the two cases. It also differs from the result in a strip. However, if both $u_1, u_2 \gg p$ and $\sqrt{u_2} - \sqrt{u_1} \gg \sqrt{p}$, it can be written as

$$G(w_1, w_2) = \left( \frac{\pi^2}{L(u_1)L(u_2)} \right)^x \exp\left[ -2\pi x_1 \left( \frac{u_2}{L(u_2)} - \frac{u_1}{L(u_1)} \right) \right] \tag{6}$$

with $L(u) = 2\sqrt{2pu}$ being the width of the system at position $u$. If in addition $u_2 - u_1 \ll (u_2 + u_1)/2 = u$, an expansion around $u$ in (6) gives back the strip result [8]. In this sense, the parabola can be considered as a strip of varying width.

If one fixes the boundary variables one can also discuss the order parameter profile. A transformation as in (3) then gives, for $w = u > 0$,

$$\langle\phi(u)\rangle = A\left(\frac{\pi^2 \tanh\zeta}{4p\zeta}\right)^x . \tag{7}$$

Thus $\langle\phi\rangle \sim u^{-x/2}$ for $u \gg p$ and there is no exponential factor in this case. The exponent, however, is still different from its value $x$ found for a corner.

One can easily generalize the treatment to boundaries which have the asymptotic form

$$v = \pm C u^\alpha \qquad \alpha < 1. \tag{8}$$

One merely has to shift the parabola into the right half-plane and then to distort it. This amounts to the replacement

$$\frac{w}{2p} \rightarrow \left(\frac{w}{2p}\right)^{2(1-\alpha)} - \frac{1}{2} \tag{9}$$

in equation (1). The quantity $C$ is then $C = (2p)^{1-\alpha}/2(1-\alpha)$. This changes the result (5) for $G$ into

$$G(w_1, w_2) = A(u_1)\frac{1}{u_2^{\alpha x}} \exp\left[\frac{-\pi x_1}{2C(1-\alpha)} u_2^{1-\alpha}\right]. \tag{10}$$

The functional form of $G$ thus varies continuously with the parameter $\alpha$ describing the boundary shape. The more this shape approaches the corner geometry ($\alpha \rightarrow 1$), the slower the exponential fall-off becomes. On the other hand, for $\alpha = 0$ the system forms a half-strip and one recovers the corresponding simple exponential decay [8,9]. For $\alpha < 0$, it has a spoon-like shape and the decay becomes very rapid in the narrow region.

We now turn to a system in the shape of figure 1($b$). Here the boundary is curved towards the outside, so that $\alpha > 1$ in equation (8). To relate it to the upper $z$-plane one has to use a different mapping, namely

$$z = \mathrm{i}\left[w^s - \left(\frac{p}{2}\right)^s\right]^{1/s} \tag{11}$$

where $s = 1 - 1/\alpha$. Asymptotically, one now has $z = \mathrm{i}w$ and the rescaling factor $|\,\mathrm{d}w/\mathrm{d}z\,|$ appearing in the transformation of $G$ becomes one. Therefore, for $w_1, w_2$ on the axis, with $u_1$ fixed and $u_2 \rightarrow \infty$, one always obtains the result of the half-plane

$$G(w_1, w_2) = A(u_1)\frac{1}{u_2^{x+x_1}}. \tag{12}$$

In this sense, this type of boundary is equivalent to a straight surface.

Finally, a system with a cut-out portion in the form of equation (8) as in figure 1($c$), can be obtained via a mapping

$$z = \mathrm{i}\left[w^s - \left(\frac{p}{2}\right)^s\right]^{1/2s} \tag{13}$$

where now $s = 1 - \alpha$ and $\alpha < 1$ again. Asymptotically, the relation therefore is, for all $\alpha$, $z = \mathrm{i}\sqrt{w}$. But with such a transformation one maps the $z$-half-plane onto the $w$-plane with a cut [2–4]. The correlation function therefore is asymptotically

$$G(w_1, w_2) = A(u_1)\frac{1}{u_2^{x+x_2}} \tag{14}$$

with the corner exponent $x_2 = \frac{1}{2}x_1$ corresponding to the cut.

The preceding results can be understood if one considers the behaviour of the boundary curve, equation (8), under a change of scale in a renormalization procedure. With $u' = u/b$, $v' = v/b$ it becomes

$$v' = \pm b^{\alpha-1}C(u')^{\alpha} \tag{15}$$

or

$$C' = b^{\alpha-1}C. \tag{16}$$

Thus, for $\alpha > 1$, $C$ grows under renormalization and the boundary curve approaches a straight line. For $\alpha = 1$, $C$ is invariant and thus a marginal variable. This explains the particular role of a corner formed by two straight lines. For $\alpha < 1$, $C$ decreases and the system approaches either a cut geometry or a one-dimensional line geometry. In the latter case, however, one has a non-ordering system and this causes the particular features of the parabolic geometry.

According to equation (16), $1/C$ may be considered as a scaling field with dimension $1 - \alpha$ (like $1/L$ in finite-size scaling). It vanishes at the half-plane fixed point. One may therefore write the following scaling ansatz for the correlations along the $u$-axis

$$G\left(u_1, u_2, \frac{1}{C}\right) = b^{-2x}G\left(\frac{u_1}{b}, \frac{u_2}{b}, \frac{b^{1-\alpha}}{C}\right). \tag{17}$$

With $b = C^{1/1-\alpha}$, one gets

$$G\left(u_1, u_2, \frac{1}{C}\right) = C^{-2x/(1-\alpha)}g\left(\frac{u_1}{L(u_1)}, \frac{u_2}{L(u_2)}\right) \tag{18}$$

where $L(u) = 2Cu^{\alpha}$ is the width of the system at $u$. Equation (6) can thereby be generalized to any value of $\alpha < 1$ with the scaling function given by

$$g(a_1, a_2) \sim (a_1 a_2)^{-x\alpha/(1-\alpha)} \exp\left[-\frac{\pi x_1}{1-\alpha}(a_2 - a_1)\right] \tag{19}$$

when $a_1, a_2 \gg 1$ and $a_2 - a_1 \gg 1$. This can also be verified explicitly. The scaling behaviour of the order parameter profile is obtained in the same way and reads

$$\langle\phi(u)\rangle = L(u)^{-x} f\left(\frac{u}{L(u)}\right) \tag{20}$$

where, according to equation (7), $\lim_{a\to\infty} f(a) = \mathrm{O}(1)$.

Relations like (18) and (20) correspond to a local formulation of finite-size scaling. One may also notice that all these scaling considerations still apply in higher dimensions.

Finally, let us address briefly the ordered state. *A priori*, it is not obvious that a system in the shape of figure 1($a$) will order at all. We have therefore studied an Ising model with parabolic shape $v = \pm C\sqrt{u}$, using the corner transfer matrix technique [10]. This means that one considers the transfer matrix connecting the spins at the upper and lower boundaries with fixed boundary condition on the right end of the system. Assuming a square lattice in the Hamiltonian limit [11] one then is led to study the following operator (describing an inhomogeneous transverse Ising chain)

$$H = -C\left[\sum_{n=1}^{N-1} \sqrt{n}\ \sigma_n^z + \lambda \sum_{n=0}^{N-1} \sqrt{n+1}\ \sigma_n^x \sigma_{n+1}^x\right] \tag{21}$$

where $\lambda^{-1}$ measures the temperature and $N$ is the size of the system along the axis. The coefficients reflect, in the sense of a continuum limit, the number of vertical and of horizontal bonds at position $n$, respectively. The transverse field vanishes at $n = 0$ due to the absence of vertical bonds for the first spin and at $n = N$ as a consequence of the boundary condition. The operator can be diagonalized in terms of fermions. The single-particle excitation energies $\epsilon_\nu = 2C\omega_\nu$ then follow from

$$n\lambda\psi_{n-1}^\nu + n(\lambda^2+1)\psi_n^\nu + n\lambda\psi_{n+1}^\nu = \omega_\nu^2\psi_n^\nu \tag{22}$$

with appropriate boundary conditions at $n = 0,\ N$. This system of equations is similar to one studied previously in a related context [12] and, as there, can be solved with Gottlieb polynomials. In the limit $N \to \infty$ one finds, for $\lambda > 1$,

$$\omega_\nu = \sqrt{(\lambda^2-1)\nu} \qquad \nu = 1, 2, 3 \cdots. \tag{23}$$

Identifying the boundaries, the magnetization at the tip of a system which is isotropic at the critical point, is given by [10,11]

$$m_0 = \langle\sigma_0^x\rangle = \prod_\nu \tanh\left(\frac{\epsilon_\nu}{2}\right). \tag{24}$$

Evaluating this near the critical point ($\lambda \gtrsim 1$) leads to

$$m_0 \sim \exp\left[\frac{-a}{C^2(\lambda-1)}\right] \tag{25}$$

where $a = 7\zeta(3)/16 \simeq 0.526$. Thus there is order, but it vanishes exponentially fast at the critical point. This reflects the difficulty to maintain it in such a geometry. We note that the argument in the exponential can be expressed as the ratio $\xi/p$ where $\xi \sim (\lambda-1)^{-1} \sim t^{-\nu}$ is the bulk correlation length.

The behaviour of the tip magnetization may also be deduced from scaling considerations. The magnetization at position $u$ along the axis satisfies

$$m\left(t, u, \frac{1}{C}\right) = b^{-x} m\left(b^{1/\nu}t, \frac{u}{b}, \frac{b^{1-\alpha}}{C}\right) \tag{26}$$

which, with $b = t^{-\nu}$, leads to

$$m\left(t, u, \frac{1}{C}\right) = t^{\beta} f\left(\frac{u}{t^{-\nu}}, \frac{t^{-\nu(1-\alpha)}}{C}\right) \tag{27}$$

where $\beta$ and $\nu$ are bulk exponents.

One may even go further assuming that, at the tip, the leading contribution to the magnetization which is induced by the bulk at a distance $D \sim (\xi/C)^{1/\alpha}$ (where the width of the system is of the order of the bulk correlation length), decays with $D$ like the correlation function in (10) when $u_1 \to 0$ and $u_2 = D \gg C^{1/1-\alpha}$. Then

$$m\left(t, 0, \frac{1}{C}\right) = m_0 \sim \exp\left(-a\frac{D^{1-\alpha}}{C(1-\alpha)}\right) \tag{28}$$

and the temperature dependence of the tip magnetization follows

$$m_0 \sim \exp\left[-a\left(\frac{t^{-\nu(1-\alpha)}}{C(1-\alpha)}\right)^{1/\alpha}\right] \tag{29}$$

in agreement with (24) for the Ising parabola with $\alpha = \frac{1}{2}$ and $\nu = 1$.

One should mention that the results (10) for the correlation function and (29) for the order parameter are quite similar to those obtained for an Ising model with bond strengths decreasing towards a free surface as $K(n) = K(\infty)(1 - A/n^y)$, $y < 1$ [13–15] with the correspondences $y \leftrightarrow \alpha$, $A \leftrightarrow C$. This can be understood qualitatively since in both cases the surface order near the critical point can only be maintained through the action of the far-away bulk portion of the system.

Finally, for an anisotropic system with correlation length exponents $\nu_{\parallel}$ (along the $u$-axis) $\neq \nu_{\perp}$, the scaling dimension of $1/C$ is changed into $1 - \alpha\nu_{\parallel}/\nu_{\perp}$. Therefore the perturbation to the half-plane geometry then becomes relevant when $\alpha < \nu_{\perp}/\nu_{\parallel}$.

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